In Chapter 2 we have shown how the governing equations for seismic wave propagation can be represented as coupled sets of first order equations in terms of the stress-displacement vector \( \mathbf{b} \). We now turn our attention to the construction of stress-displacement fields in stratified media.

We start by considering a uniform medium for which we can make an unambiguous decomposition of the wavefield into up and downgoing parts. We then treat the case where the seismic properties vary smoothly with depth. Extensions of the approach used for the uniform medium run into problems at the turning points of \( P \) or \( S \) waves. These difficulties can be avoided by working with uniform approximations based on Airy functions, which behave asymptotically like up and downgoing waves.

### 3.1 A uniform medium

An important special case of a ‘stratified’ medium is a uniform medium, for which we can split up a seismic disturbance into its \( P \) and \( S \) wave contributions. This separation is preserved under the Fourier-Hankel transformation (2.19) and the cylindrical waves for each wave type can be further characterised as up or downgoing by the character of their dependence on the \( z \) coordinate.

We will now show how to relate the stress-displacement vector \( \mathbf{b} \) to the up and downgoing waves in a uniform medium, and then use this relation to illustrate the fundamental and propagator matrices introduced in Section 2.2.

For a cylindrical wave with frequency \( \omega \), slowness \( p \) and angular order \( m \), we introduce a transformation which connects the stress-displacement vector \( \mathbf{b} \) to a new vector \( \mathbf{v} \)

\[
\mathbf{b} = \mathbf{Dv},
\]

and try to choose the matrix \( \mathbf{D} \) to give a simple form for the evolution of \( \mathbf{v} \) with \( z \). In a source-free region \( \mathbf{v} \) must satisfy

\[
\partial_z (\mathbf{Dv}) = \omega \mathbf{A}(p, z)\mathbf{Dv},
\]
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and so

\[ \partial_z \mathbf{v} = [\omega \mathbf{D}^{-1} \mathbf{A} - \mathbf{D}^{-1} \partial_z \mathbf{D}] \mathbf{v}. \]  
\[ (3.3) \]

If we choose \( \mathbf{D} \) to be the local eigenvector matrix for \( \mathbf{A}(p, z) \), the first element on the right hand side of (3.3) reduces to diagonal form,

\[ \omega \mathbf{D}^{-1} \mathbf{A} = i \omega \mathbf{A}, \]
\[ (3.4) \]
where \( i \mathbf{A} \) is a diagonal matrix whose entries are the eigenvalues of \( \mathbf{A} \). From the explicit forms of the coefficient matrices in (2.24) and (2.25) we find that for \( P-SV \) waves,

\[ \mathbf{A}_P = \text{diag}[-q_\alpha, -q_\beta, q_\alpha, q_\beta], \]
\[ (3.5) \]
and for \( SH \) waves

\[ \mathbf{A}_H = \text{diag}[-q_\beta, q_\beta]; \]
\[ (3.6) \]
where

\[ q_\alpha = (\alpha^2 - p^2)^{1/2}, \quad q_\beta = (\beta^2 - p^2)^{1/2}, \]
\[ (3.7) \]
are the vertical slownesses for \( P \) and \( S \) waves for a horizontal slowness \( p \). The choice of branch cuts for the radicals \( q_\alpha, q_\beta \) will normally be

\[ \text{Im} \omega q_\alpha \geq 0, \quad \text{Im} \omega q_\beta \geq 0, \]
\[ (3.8) \]
the frequency factor enters from (3.4).

In a uniform medium the coefficient matrix \( \mathbf{A} \) is constant and so the eigenvector matrix \( \mathbf{D} \) is independent of \( z \), with the result that \( \mathbf{D}^{-1} \partial_z \mathbf{D} \) vanishes. The vector \( \mathbf{v} \) is then governed by the differential equation

\[ \partial_z \mathbf{v} = i \omega \mathbf{A} \mathbf{v}, \]
\[ (3.9) \]
with a solution

\[ \mathbf{v}(z) = \exp[i \omega (z - z_0) \mathbf{A}] \mathbf{v}(z_0) = \mathbf{Q}(z, z_0) \mathbf{v}(z_0), \]
\[ (3.10) \]
in terms of a ‘wave-propagator’ \( \mathbf{Q} \), which depends on the difference between the current depth \( z \) and the reference level \( z_0 \). The exponential of a diagonal matrix is a further diagonal matrix with exponential entries and so for \( P-SV \) waves,

\[ \mathbf{Q}_P(h, 0) = \text{diag}[e^{-i \omega q_\alpha h}, e^{-i \omega q_\beta h}, e^{i \omega q_\alpha h}, e^{i \omega q_\beta h}]; \]
\[ (3.11) \]
and for \( SH \) waves,

\[ \mathbf{Q}_H(h, 0) = \text{diag}[e^{-i \omega q_\beta h}, e^{i \omega q_\beta h}]. \]
\[ (3.12) \]
With our convention that \( z \) increases with increasing depth, these exponentials correspond to the phase increments that we would expect for the propagation of upward and travelling \( P \) and \( S \) waves through a vertical distance \( h \). For example,
3.1 A uniform medium

Suppose that we have a plane $S$ wave travelling downward at an angle $j$ to the $z$ axis, then

$$ p = \sin j/\beta, \quad q_\beta = \cos j/\beta, \quad (3.13) $$

and the phase difference we would expect to be introduced in traversing a depth interval $h$ is

$$ \exp[i\omega h \cos j/\beta] = \exp[i\omega q_\beta h]; \quad (3.14) $$

for upgoing waves we would have the inverse of (3.14).

From (3.10) the wavevector $\mathbf{v}$ at $z$ is just a phase shifted version of its value at $z_0$ and we may identify the elements of $\mathbf{v}$ with up or downgoing $P$ and $S$ waves by (3.11), (3.12). For $P$-$SV$ waves we set

$$ \mathbf{v}_P = [P_U, S_U, P_D, S_D]^T, \quad (3.15) $$

where $P, S$ are associated with $P$ and $SV$ propagation and the suffices $U, D$ represent up and downgoing waves; for $SH$ waves we denote the elements by $H$, so that

$$ \mathbf{v}_H = [H_U, H_D]^T. \quad (3.16) $$

We may summarise the behaviour of the wavevector $\mathbf{v}$ by introducing partitions corresponding to up and downgoing waves

$$ \mathbf{v} = [\mathbf{v}_U, \mathbf{v}_D]^T. \quad (3.17) $$

When the horizontal slowness becomes larger than the inverse wavespeeds $\alpha^{-1}, \beta^{-1}$ the corresponding radicals $q_\alpha, q_\beta$ become complex. With our choice of radical, in a perfectly elastic medium with $p > \beta^{-1},$

$$ \exp[i\omega q_\beta z] = \exp[-\omega |q_\beta| z], \quad (3.18) $$

and so downgoing waves $\mathbf{v}_D$ in the propagating regime ($p < \beta^{-1}$) map to evanescent waves which decay with depth. This property extends to a dissipative medium but is not as easily illustrated. In a similar way the upgoing waves $\mathbf{v}_U$ map to evanescent waves which increase exponentially with increasing depth $z$.

From the initial value solution for the wavevector $\mathbf{v}$ (3.10) we can construct the initial value solution for the stress-displacement $\mathbf{b}$, in the form

$$ \mathbf{b}(z) = \mathbf{D} \exp[i\omega (z - z_0)|\mathbf{A}|\mathbf{D}^{-1}\mathbf{b}(z_0), \quad (3.19) $$

and so from (2.77) we may recognise the propagator for the uniform medium as

$$ \mathbf{P}(z, z_0) = \exp[i\omega (z - z_0)|\mathbf{A}|\mathbf{D}^{-1} = \mathbf{D} \exp[i\omega (z - z_0)|\mathbf{A}|\mathbf{D}^{-1}. \quad (3.20) $$

We have been able to simplify the calculation of the matrix exponential by the use of the similarity transformation provided by $\mathbf{D}$. From the representation of the propagator matrix in terms of a fundamental stress-displacement matrix $\mathbf{B}$, (2.64), we can recognise a fundamental matrix for the uniform medium

$$ \mathbf{B}(z) = \mathbf{D} \exp[i\omega (z - z_{ref})|\mathbf{A}|, \quad (3.21) $$
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where $z_{\text{ref}}$ is the reference level for the phase of the $P$ and $S$ wave elements. The eigenvector matrix $D$ may now be seen to be this fundamental matrix evaluated at the reference level $z_{\text{ref}}$, and thus its columns may be identified as ‘elementary’ stress-displacement vectors corresponding to the different wavetypes. For $P$-$SV$ waves

$$D_P = [\epsilon_\alpha b^P_{UI}, \epsilon_\beta b^S_{UI}, \epsilon_\alpha b^P_{UD}, \epsilon_\beta b^S_{UD}],$$

(3.22)

where

$$b^P_{UI,D} = [\pm iq_\alpha, p, \rho(2\beta^2p^2 - 1), \pm 2i\rho \beta^2pq_\alpha]^T,$$

$$b^S_{UI,D} = [p, \mp iq_\beta, \mp 2i\rho \beta^2pq_\beta, \rho(2\beta^2p^2 - 1)]^T,$$

(3.23)

and we take the upper sign for the upgoing elements and the lower for downgoing elements. For $SH$ waves

$$D_H = [\epsilon_H b^H_{UI}, \epsilon_H b^H_{UD}],$$

(3.24)

with

$$b^H_{UI,D} = [\beta^{-1}, \mp i\rho \beta q_\beta]^T.$$

(3.25)

We have chosen the scaling to give comparable dimensionality to corresponding elements of $b^P$, $b^S$, $b^H$; the $SH$ waveslowness $\beta^{-1}$ appears in (3.25) in a similar role to the horizontal slowness in (3.23). We have a free choice of the scaling parameters $\epsilon_\alpha$, $\epsilon_\beta$, $\epsilon_H$ and we would like the quantities $P_{UI}$, $S_{UI}$, $H_{UI}$, etc. to have comparable meanings. It is convenient to normalise these $b$ vectors so that, in a perfectly elastic medium, each of them carries the same energy flux in the $z$ direction for a propagating wave.

The energy flux crossing a plane $z = \text{const}$ is given by an area integral of the scalar product of the velocity and the traction on the plane

$$E = 2\pi \int_0^r dr r \sum_j w_j \tau_{jz}. $$

(3.26)

At a frequency $\omega$ we may represent the areal flux, averaged over a cycle in time, as

$$\langle E \rangle = 2\pi \int_0^r dr r \frac{-i\omega}{4} \sum m [\tilde{w}.\tilde{t}^* - \tilde{w}^*.\tilde{t}],$$

(3.27)

where the overbars denote a Fourier transform with respect to time. We may now use the vector harmonic expansion for displacement and tractions (2.55) and the orthonormality properties of the vector harmonics (2.57-(2.58)) to evaluate (3.27) as

$$\langle E \rangle = \frac{1}{2\pi} \int_0^\infty dk k \sum_m \frac{-i\omega^2}{4} [U^*P + VS^* + WT^* - U^*P - V^*S - W^*T].$$

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For an individual cylindrical wave we can therefore construct measures of the associated energy flux: for $P$-$SV$ waves we take

$$\Upsilon_p(b) = i[UP^* + VS^* - U^*P - V^*S],$$

(3.29)

which is just $ib^{T_s}N b$ (cf. 2.37), and for $SH$ waves

$$\Upsilon_H(b) = i[W^T - W^*T].$$

(3.30)

Let us now consider the vector $\epsilon_\alpha b^P_D$ for a downgoing propagating $P$ wave in a perfectly elastic medium, i.e. $q_\alpha$ real, then

$$\Upsilon_p = |\epsilon_\alpha|^2 2\rho q_\alpha.$$  

(3.31)

A convenient normalisation is to take

$$\epsilon_\alpha = (2\rho q_\alpha)^{-1/2},$$

and the actual energy flux associated with $\epsilon_\alpha b^P_D$ is $\omega^2/4$. We may make a corresponding choice for the normalisations for both the $SV$ and $SH$ elements by choosing

$$\epsilon_\beta = \epsilon_H = (2\rho q_\beta)^{-1/2}.$$  

(3.33)

Thus for propagating $P$ and $S$ waves

$$\Upsilon(\epsilon_\alpha b^P_D) = \Upsilon(\epsilon_\beta b^S_D) = \Upsilon(\epsilon_\beta b^H_D) = 1,$$

$$\Upsilon(\epsilon_\alpha b^U_D) = \Upsilon(\epsilon_\beta b^S_U) = \Upsilon(\epsilon_\beta b^H_U) = -1.$$  

(3.34)

Although we have constructed $\epsilon_\alpha, \epsilon_\beta$ for a perfectly elastic medium, we will use the normalisations (3.32-3.33) in both the propagating and evanescent regimes.

For evanescent waves in a perfectly elastic medium, $q_\alpha$ and $q_\beta$ will be pure imaginary and so $\epsilon_\alpha$ and $\epsilon_\beta$ become complex and then, for example,

$$\Upsilon(\epsilon_\beta b^S_D) = 0$$

(3.35)

confirming that evanescent waves carry no energy flux in the $z$ direction.

We have already noted that the eigenvector matrix $D$ is a special case of a fundamental matrix, and we may display its role as a transformation by writing $D$ in partitioned form

$$D = \begin{bmatrix} m_U & m_D \\ n_U & n_D \end{bmatrix}.$$  

(3.36)

The partition $m_U$ transforms the upper elements $v_U$ of the wavevector into displacements and $m_D$ generates displacements from $v_D$. The partitions $n_U, n_D$ generate stresses from $v_U$ and $v_D$. Such a structure will occur in all cases including
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full anisotropy. For P-SV waves we have from (3.23)

\[
\begin{align*}
\mathbf{m}_{U,D} &= \begin{bmatrix}
\mp i q_\alpha e_\alpha & p e_\beta \\
 p e_\alpha & \mp i q_\beta e_\beta
\end{bmatrix} \\
\mathbf{n}_{U,D} &= \begin{bmatrix}
\rho (2\beta^2 p^2 - 1) e_\alpha & \mp 2 i \rho \beta^2 p q_\beta e_\beta \\
\mp 2 i \rho \beta^2 p q_\alpha e_\alpha & \rho (2\beta^2 p^2 - 1) e_\beta
\end{bmatrix}
\end{align*}
\]

and for SH waves

\[
\begin{align*}
\mathbf{m}_{U,D} &= \beta^{-1} e_\beta, \\
\mathbf{n}_{U,D} &= \mp i \rho \beta q_\beta e_\beta.
\end{align*}
\]

From these partitioned forms we can construct the propagation invariants \( \langle \mathbf{m}_U, \mathbf{m}_D \rangle \), (2.68), and in each case

\[
\mathbf{m}_U, \mathbf{m}_D = \mp i \mathbf{I}.
\]

This means that we have a particularly simple closed form inverse for \( \mathbf{D} \) via the representation (2.74)

\[
\mathbf{D}^{-1} = \frac{1}{\rho} \begin{bmatrix}
-\mathbf{n}_D^T & \mathbf{m}_D^T \\
\mathbf{n}_U^T & -\mathbf{m}_U^T
\end{bmatrix}.
\]

With these expressions for the eigenvector matrix \( \mathbf{D} \) and its inverse, we may now use (3.20) to construct expressions for the stress-displacement propagator in the uniform medium. For P-SV waves

\[
\mathbf{P}_P(h,0) = \mathbf{D}_P \text{diag}[e^{-i \omega q_\alpha h}, e^{-i \omega q_\beta h}, e^{i \omega q_\alpha h}, e^{i \omega q_\beta h}] \mathbf{D}_P^{-1},
\]

and so the partitions \( \mathbf{P}_{WW}, \mathbf{P}_{WT}, \mathbf{P}_{TW}, \mathbf{P}_{TT} \) of the propagator are given by

\[
\begin{align*}
\mathbf{P}_{WW} &= \begin{bmatrix}
2\beta^2 p^2 C_\beta - \Gamma C_\alpha & -p[2\beta^2 q_\alpha^2 S_\alpha + \Gamma S_\beta] \\
-p[\Gamma S_\alpha + 2\beta^2 q_\beta^2 S_\beta] & 2\beta^2 p^2 C_\alpha - \Gamma C_\beta
\end{bmatrix}, \\
\mathbf{P}_{WT} &= \rho^{-1} \begin{bmatrix}
q_\alpha^2 S_\alpha + p^2 S_\beta & p[C_\alpha - C_\beta] \\
p[C_\beta - C_\alpha] & p^2 S_\alpha + q_\beta^2 S_\beta
\end{bmatrix}, \\
\mathbf{P}_{TW} &= -\rho \begin{bmatrix}
4\beta^4 p^2 q_\alpha^2 S_\beta + \Gamma^2 S_\alpha & p\beta^2 \Gamma(C_\alpha - C_\beta) \\
p\beta^2 \Gamma(C_\beta - C_\alpha) & 4\beta^4 p^2 q_\alpha^2 S_\alpha + \Gamma^2 S_\beta
\end{bmatrix}, \\
\mathbf{P}_{TT} &= \begin{bmatrix}
2\beta^2 p^2 C_\beta - \Gamma C_\alpha & -p[2\beta^2 q_\alpha^2 S_\alpha + \Gamma S_\beta] \\
p[\Gamma S_\beta + 2\beta^2 q_\beta^2 S_\alpha] & 2\beta^2 p^2 C_\alpha - \Gamma C_\beta
\end{bmatrix},
\end{align*}
\]

where

\[
C_\alpha = \cos \omega q_\alpha h, \quad C_\beta = \cos \omega q_\beta h, \\
S_\alpha = q_\alpha^{-1} \sin \omega q_\alpha h, \quad S_\beta = q_\beta^{-1} \sin \omega q_\beta h,
\]

and

\[
\Gamma = 2\beta^2 p^2 - 1.
\]

The SH wave propagator is rather simpler

\[
\mathbf{P}_H(h,0) = \begin{bmatrix}
C_\beta & (\rho \beta^2)^{-1} S_\beta \\
-\rho \beta^2 q_\beta^2 S_\beta & C_\beta
\end{bmatrix}.
\]
3.2 A smoothly varying medium

In fact this expression may be easily constructed by summing the matrix exponential series (2.85). The inverses of the propagators may be found from (2.80) and for (3.42) and (3.44) we may verify the inverse propagator relation (2.90).

These uniform layer propagators are identical to the layer matrices of Haskell (1953) although they have been derived via a different route. We have followed Dunkin (1965) and diagonalised $A$ via the eigenvector matrix $D$, but other choices are possible and lead to the same result. For example Hudson (1969a) describes a transformation to variables $P_U \pm P_D$ (in our notation) and this is closely related to the original treatment of Haskell. Hudson is able to calculate the exponential of his transformed matrix by summing the series (2.85), since direct exponentiation is only convenient for a diagonal matrix.

In our construction of the stress-displacement propagator via (3.20) we have split the wavefield in the uniform medium into its component parts via $D^{-1}$. We have then added in the phase increments for the separate up and downgoing $P$ and $S$ wave contributions for a depth interval $h$ and finally reconstituted displacements and stresses via the matrix $D$.

With the aid of the expressions (3.36) and (3.40) for $D$ and its inverse, we can represent the uniform layer propagator as a sum of upgoing and downgoing contributions

$$P(h,0) = i \begin{bmatrix} -m_U E_U n_U^T & m_U E_U m_U^T \\ -n_U E_U n_U^T & n_U E_U m_U^T \end{bmatrix} + i \begin{bmatrix} m_U E_D n_U^T & -m_U E_D m_U^T \\ n_U E_D n_U^T & -n_U E_D m_U^T \end{bmatrix},$$

(3.45)

where the diagonal matrix $E_D$ is the phase income for downgoing waves, e.g., for $P$-$SV$ waves

$$E_D = \text{diag\big[} e^{i\omega q_a h}, e^{i\omega q_b h} \big],$$

(3.46)

and $E_U = E_D^{-1}$.

### 3.2 A smoothly varying medium

In many cases of interest we need to consider the propagation of seismic waves in continuously stratified regions; as for example, the wavespeed gradients due to compaction in a sedimentary sequence or the velocity profile in the Earth’s mantle. Although we may simulate a gradient by a fine cascade of uniform layers, we would like to make a direct construction of a fundamental stress-displacement matrix $B$ so that we may define the propagator matrix and establish the reflection properties of the medium.

As an extension of the treatment for a uniform medium we consider a local eigenvector transformation (3.1) at each level $z$ so that $b(z) = D(z)v(z)$. The wavevector $v$ will then be governed by the evolution equation

$$\partial_z v = [i\omega A - D^{-1} \partial_z D] v,$$

(3.47)

and the presence of the matrix $\Delta = -D^{-1} \partial_z D$ will introduce coupling between
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the elements of \( v \) which we would characterise as up or downgoing in a uniform medium. The normalisation of the columns of \( D \) to constant energy flux in the \( z \) direction means that the diagonal elements of \( \Delta \) vanish. A wave quantity such as \( P \) would therefore modified by loss to, or gain from, other components, arising from the nature of the variation of the elastic properties with depth. The coupling matrix \( \Delta \) depends on the gradients of the vertical slownesses \( q_\alpha, q_\beta \) and the elastic parameters (Chapman, 1974a).

For \( SH \) waves the coupling matrix is symmetric with only off-diagonal elements

\[
\Delta_H = -D_H^{-1} \partial_z D_H = \begin{bmatrix} 0 & \gamma_H \\ \gamma_H & 0 \end{bmatrix}
\]

(3.48)

where

\[
\gamma_H = \frac{1}{2} \partial_z q_\beta / q_\beta + \frac{1}{2} \partial_z \mu / \mu,
\]

(3.49)
in terms of the shear modulus \( \mu = \rho \beta^2 \). The coefficient \( \gamma_H \) determines the transfer between \( H_U \) and \( H_D \). For \( P-SV \) waves

\[
\Delta_P = -D_P^{-1} \partial_z D_P = \begin{bmatrix} 0 & -i\gamma_T & -i\gamma_R \\ -i\gamma_T & 0 & \gamma_S \\ i\gamma_R & \gamma_S & i\gamma_T \end{bmatrix},
\]

(3.50)

with

\[
\gamma_T = p(q_\alpha q_\beta)^{-1/2} \left[ \beta^2 (p^2 + q_\alpha q_\beta) \partial_z \mu / \mu - \frac{1}{2} \partial_z \rho / \rho \right],
\]

\[
\gamma_R = p(q_\alpha q_\beta)^{-1/2} \left[ \beta^2 (p^2 - q_\alpha q_\beta) \partial_z \mu / \mu - \frac{1}{2} \partial_z \rho / \rho \right],
\]

\[
\gamma_A = 2 \beta^2 (p^2) \partial_z \mu / \mu - \frac{1}{2} \partial_z \rho / \rho,
\]

and

\[
\gamma_P = \gamma_A + \frac{1}{2} \partial_z q_\alpha / q_\alpha,
\]

\[
\gamma_S = \gamma_A + \frac{1}{2} \partial_z q_\beta / q_\beta.
\]

(3.51)

It is interesting to note that the \( P \) wavespeed appears only indirectly through the slowness \( q_\alpha \). The coefficient \( \gamma_T \) determines the rate of conversion of \( P \) to \( S \) or \textit{vice versa} for elements of the same sense of propagation (e.g. \( P_U \) and \( S_U \)); \( \gamma_R \) determines the rate of conversion for elements of different sense (e.g. \( P_U \) and \( S_D \)). \( \gamma_P \) governs the rate of transfer between \( P_U \) and \( P_D \); and \( \gamma_S \) has a similar role for \( S_U \) and \( S_D \). If we consider a thin slab of material the elements \( \gamma \) are closely related to the reflection and transmission coefficients for the slab (see Section 5.5). If the elements of the coupling matrix \( \Delta \) are small compared with the diagonal terms in \( \omega \mathbf{A} \), we may construct a good approximation to a fundamental stress-displacement matrix as

\[
\mathbf{B}_0(z) = \mathbf{D}(z) \exp[i\omega \int_{z_{\text{ref}}}^z \mathbf{A}(\zeta)] = \mathbf{D}(z) \mathbf{E}(z).
\]

(3.52)

When we recall the normalisation implicit in \( \mathbf{D}(z) \) we find that \( \mathbf{B}_0 \) corresponds to a WKBJ solution assuming independent propagation of each wave type. The
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phase integral allows for the variation of the vertical slownesses with depth and \( \epsilon_\alpha, \epsilon_\beta \) maintain constant energy flux for each wave-element \( P_U, S_U \) etc. This WKBJ solution is just what would be predicted to the lowest order in ray theory, using the ideas of energy conservation along a ray tube and phase delay.

The approximation (3.52) makes no allowance for the presence of the coupling matrix \( \Delta \). In any better approximation we wish to retain the phase terms in \( \mathcal{E}(z) \), arising from the \( i\omega \Lambda \) diagonal elements in (3.47). These diagonal elements are a factor of \( \omega \) larger than the coupling terms in \( \Delta \) and so should dominate at high frequencies. This led Richards (1971) to suggest an asymptotic expansion in inverse powers of \( \omega \),

\[
B(z) = D(z) [I + \sum_{r=1}^{\infty} X_r (i\omega)^r] \mathcal{E}(z). \tag{3.53}
\]

The matrices \( X_r \) in this ‘ray-series’ expansion are determined recursively. When we substitute (3.53) into the equation (2.64) for \( B \) and examine the equations for each power of \( \omega \) in turn, we find that \( X_r \) depends on \( X_{r-1} \) as

\[
[X_r \Lambda - \Lambda X_r] = \partial_z X_{r-1} + \Delta X_{r-1}. \tag{3.54}
\]

Starting with \( X_0 = I \), this commutator relation may be solved to find the \( X_r \).

An alternative approach used by Chapman (1976) and Richards & Frasier (1976) is to look for a solution in the form

\[
B(z) = D(z) \mathcal{E}(z) \Psi(z), \tag{3.55}
\]

and then, since \( \partial_z \mathcal{E} = i\omega \Lambda \mathcal{E} \), the correction term \( \Psi \) satisfies

\[
\partial_z \Psi(z) = \mathcal{E}^{-1}(z) \Delta(z) \mathcal{E}(z) \Psi(z), \tag{3.56}
\]

and a solution may be constructed in a similar way to our treatment of the propagator (2.85) as the series

\[
\Psi(z) = I + \int_z^{z_{\text{ref}}} d\zeta \mathcal{E}^{-1}(\zeta) \Delta(\zeta) \mathcal{E}(\zeta) + \ldots. \tag{3.57}
\]

The successive integrals in (3.57) can be identified by their phase behaviour with multiple reflections within the varying medium.

The two approximation schemes we have described are of greatest utility when only the first correction terms to \( B_0 \) are significant. This requires that all the elements in the coupling matrix \( \Delta \) must be small, but even for velocity models with only slow variation with depth the elements of \( \Delta \) can be large. The matrix \( \Delta \) becomes singular when one of the vertical slownesses \( q_\alpha \) or \( q_\beta \) vanishes, i.e. when \( \alpha^{-1}(Z_\alpha) = p \) or \( \beta^{-1}(Z_\beta) = p \). This level in a perfectly elastic medium, separates the region where \( q_\alpha \) is real and we have propagating waves, from the evanescent region where \( q_\alpha \) is imaginary. If the \( P \) wavespeed increases with depth, the turning level \( Z_\alpha \) will correspond to the level at which an initially downgoing ray with inclination to the vertical \( \sin^{-1}[\alpha(z)/p] \) will be travelling horizontally.
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In this ray picture the effect of wavefront curvature will be sufficient to turn the ray back up towards the surface. In the neighbourhood of this turning level any attempt to separate the wavefield into specifically downgoing and upgoing elements is imposing an artificial structure, this is reflected by the singularity in $\Delta$ at the level of total reflection $Z_\alpha$.

In a weakly attenuative medium there is no real turning level at which $q_\alpha$ vanishes, nevertheless the coefficients in $\Delta$ become very large in the region where $\Re \ q_\alpha$ is very small, and it is still inappropriate to make a decomposition into up and downgoing waves in this zone.

Well away from a turning level, in either the propagating or evanescent regimes, the WKBJ approximation (3.52), supplemented perhaps by a single correction term as in (3.57), will provide a good description of the propagation process in a smoothly varying medium. In particular this approach is useful for near-vertical incidence ($p$ small) when we are interested in the waves returned by the structure.

In the following section we discuss an alternative construction for a fundamental $B$ matrix in a smoothly varying medium which avoids the singular behaviour at the turning levels and leads to a uniform approximation.

3.3 Uniform approximations for a smoothly varying medium

The difficulties with the eigenvector matrix decomposition are associated with the singularities at the turning points of $P$ and $S$ waves. At a $P$ wave turning level the coefficient $\gamma_P$ linking the $P_U$ and $P_D$ elements has a $q_\alpha^{-1}$ singularity at $Z_\alpha$ which is the main difficulty. The integrable $q_\alpha^{-1/2}$ singularity in the coefficients $\gamma_R$, $\gamma_T$ which control the conversion from $P$ to $S$ waves causes no major problems.

A simple example which exhibits turning point behaviour is the linear slowness profile for a scalar wave. A solution which represents both the behaviour in the propagating regime and the exponential decay below the turning level was given by Gans (1915). This solution may be written in terms of an Airy function $Ai(x)$. Langer (1937) recognised that by a mapping of the argument of the Airy function a uniform asymptotic solution across a turning point can be found for a general monotonic slowness distribution.

For $SH$ waves the Langer approach can be used with little modification. We look for a fundamental matrix $B$, in the form

$$B = CU,$$  \hspace{1cm} (3.58)

and then $U$ will satisfy

$$\partial_zU = [\omega C^{-1}AC - C^{-1}\partial_zC]U.$$  \hspace{1cm} (3.59)
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We choose the new transformation matrix \( C_H \) so that \( C_H^{-1} A_H C_H \) has only off-diagonal elements

\[ H_\beta = C_H^{-1} A_H C_H = \begin{bmatrix} 0 & p \\ -q_\beta^2/p & 0 \end{bmatrix}, \quad (3.60) \]

and now rather than find the matrix exponential of (3.60) we will seek a matrix \( E_\beta \) which provides a good asymptotic representation of the solution of (3.59) at high frequencies. A suitable transformation matrix \( C_H \) is

\[ C_H = (\rho p)^{-1/2} \begin{bmatrix} \beta^{-1} & 0 \\ 0 & \rho p \beta \end{bmatrix}, \quad (3.61) \]

which provides a rescaling of the stress and displacement elements. \( C_H \) does not depend on the radical \( q_\beta \), with the result that the new coupling matrix \(-C_H^{-1} \partial_z C_H\) depends only on the elastic parameter gradients

\[ -C_H^{-1} \partial_z C_H = \begin{bmatrix} \frac{1}{2} \partial_z \mu/\mu & 0 \\ 0 & \frac{1}{2} \partial_z \mu/\mu \end{bmatrix}, \quad (3.62) \]

and is well behaved at the turning level for \( SH \) waves.

When we consider the \( P-SV \) wave system we are only able to apply the Langer approach to one wave type at a time, and so we must make a transformation as in (3.58) to bring \( C_P^{-1} A_P C_P \) into block diagonal form where the entry for each wave type has the structure (3.60). Thus we seek

\[ H = C_P^{-1} A_P C_P = \begin{bmatrix} H_\alpha & 0 \\ 0 & H_\beta \end{bmatrix}, \quad (3.63) \]

and guided by the work of Chapman (1974b) and Woodhouse (1978) we take

\[ C_P = (\rho p)^{-1/2} \begin{bmatrix} 0 & p & p & 0 \\ p & 0 & 0 & p \\ \rho(2\beta^2 p^2 - 1) & 0 & 0 & 2\rho\beta^2 p^2 \\ 0 & 2\rho\beta^2 p^2 & \rho(2\beta^2 p^2 - 1) & 0 \end{bmatrix}, \quad (3.64) \]

As in the \( SH \) case we have avoided the slownesses \( q_\alpha, q_\beta \) but there is no longer such a simple relation between \( B \) and \( U \). It is interesting to note that we can construct the columns of \( C_P \) by taking the sum and difference of the columns of \( D_P \) and rescaling; this suggests that we may be able to make a standing wave interpretation of \( U \). The \( P-SV \) coupling terms are given by

\[ -C_P^{-1} \partial_z C_P = \begin{bmatrix} \gamma_A & 0 & 0 & -\gamma_C \\ 0 & -\gamma_A & -\gamma_B & 0 \\ 0 & \gamma_C & \gamma_A & 0 \\ \gamma_C & 0 & 0 & -\gamma_A \end{bmatrix}, \quad (3.65) \]
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where $\gamma_A$ has already been defined in (3.47) and controls the rate of change of the $P$ and $S$ wave coefficients. The off-diagonal terms in (3.61)

$$
\gamma_B = 2\beta^2 p^2 \partial_2 \mu / \mu - \partial_2 \rho / \rho, \quad \gamma_C = 2\beta^2 p^2 \partial_2 \mu / \mu,
$$

(3.66)

lead to cross-coupling between $P$ and $S$ elements.

We now introduce a ‘phase-matrix’ $\hat{E}_\beta$ which matches the high frequency part of (3.59). Following Woodhouse (1978) we construct $\hat{E}_\beta$ from Airy function entries. The two linearly independent Airy functions $Ai(x), Bi(x)$ are solutions of the equation

$$
\partial^2 y / \partial x^2 - xy = 0
$$

(3.67)

and so, e.g., the derivative of $Ai'$ is just a multiple of $Ai$. Thus a matrix with entries depending on Airy functions and their derivatives will match the off-diagonal high frequency part of (3.55). In a slightly dissipative medium we take

$$
\hat{E}_\beta(\omega, p, z) = \pi^{1/2} \left[ s_\beta \omega^{1/6} e^{1/2} Bi(-\omega^{2/3} \phi_\beta) \quad s_\beta \omega^{1/6} e^{1/2} Ai(-\omega^{2/3} \phi_\beta) \right]
$$

$$
\left[ \omega^{-1/6} e^{-1/2} Bi'(-\omega^{2/3} \phi_\beta) \quad \omega^{-1/6} e^{-1/2} Ai'(-\omega^{2/3} \phi_\beta) \right],
$$

(3.68)

with

$$
s_\beta = -\partial_2 \phi_\beta / |\partial_2 \phi_\beta|, \quad \tau_\beta = p / |\partial_2 \phi_\beta|.
$$

(3.69)

The argument of the Airy functions is chosen so that the derivative of $E_\beta$ can be brought into the same form as $H_\beta$ (3.60), so we require

$$
\phi_\beta(\partial_2 \phi_\beta)^2 = q_\beta^2 = \beta^{-2} - p^2.
$$

(3.70)

The solution for $\phi_\beta$ can be written as

$$
\omega^{2/3} \phi_\beta = \text{sgn}[\text{Re} q_\beta^2] \left[ \frac{3}{2} \omega \tau_\beta \right]^{2/3}
$$

(3.71)

where

$$
\omega \tau_\beta = \int_z^{z_\beta} d\zeta \omega q_\beta(\zeta), \quad \text{Re} q_\beta^2 > 0,
$$

$$
= \int_z^{z_\beta} d\zeta i \omega q_\beta(\zeta), \quad \text{Re} q_\beta^2 < 0,
$$

(3.72)

for positive frequency $\omega$; we have here made use of our choice of branch cut (3.8) \{Im $\omega q_\beta \geq 0$\} for the radical $q_\beta$. We take the principal value of the $^2/3$ power in (3.71). In a perfectly elastic medium these rather complex forms simplify to

$$
\phi_\beta = \text{sgn}(q_\beta^2) \left[ \frac{3}{2} \tau_\beta \right]^{2/3},
$$

(3.73)

$$
s_\beta = -\text{sgn}(\partial_2 \phi_\beta), \quad \tau_\beta = \int_z^{z_\beta} d\zeta |q_\beta(\zeta)|.
$$

(3.74)

We take $z_\beta$ to be some convenient reference level. In a perfectly elastic medium,
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when a turning point exists i.e. \( q_\beta(Z_\beta) = 0 \) at a real depth \( Z_\beta \), then \( \phi_\beta \) will be regular and unique with the choice \( z_\beta = Z_\beta \). For dissipative media, the location of the root \( q_\beta = 0 \) has to be found by analytically continuing the wavespeed profile to complex depth. If we then take this complex value for \( z_\beta \), the integral for \( \tau_\beta \) will be a contour integral. For small dissipation \( Q_\beta^{-1} \), it is simpler to take \( z_\beta \) to be the depth at which \( \text{Re } q_\beta = 0 \). If the slowness \( p \) is such that no turning point occurs for \( S \) waves in the region of interest, then any choice of \( z_\beta \) may be taken so that \( \phi_\beta \) is non-unique. In order that the character of the phase matrix \( \hat{E}_\beta \) should correspond to the physical nature of the wave propagation, it is often desirable to extrapolate \( \beta(z) \) so that a turning point \( Z_\beta \) is created. This artificial turning level is then used as the reference to reckon \( \tau_\beta \).

To simplify subsequent discussion we will use an abbreviated form for the elements of \( \hat{E}_\beta \) (Kennett & Woodhouse, 1978; Kennett & Illingworth, 1981)

\[
\hat{E}_\beta = \begin{bmatrix}
\scriptstyle s_\beta B_j(\omega \tau_\beta) & \scriptstyle s_\beta A_j(\omega \tau_\beta) \\
\scriptstyle B_k(\omega \tau_\beta) & \scriptstyle A_k(\omega \tau_\beta)
\end{bmatrix}.
\]

(3.75)

The inverse of \( \hat{E}_\beta \) is readily constructed by using the result that the Wronskian of \( A_i \) and \( B_i \) is \( \pi^{-1} \) and has the form

\[
\hat{E}_\beta^{-1} = \begin{bmatrix}
\scriptstyle -s_\beta A_k(\omega \tau_\beta) & \scriptstyle A_j(\omega \tau_\beta) \\
\scriptstyle s_\beta B_k(\omega \tau_\beta) & \scriptstyle -B_j(\omega \tau_\beta)
\end{bmatrix}.
\]

(3.76)

This ‘phase matrix’ \( \hat{E}_\beta \) satisfies

\[
\partial_z \hat{E}_\beta = \left[ \omega H_\beta + \partial_z \Phi_\beta \Phi_\beta^{-1} \right] \hat{E}_\beta;
\]

(3.77)

the diagonal matrix \( \partial_z \Phi_\beta \Phi_\beta^{-1} \) depends on the Airy function argument \( \phi_\beta \)

\[
\partial_z \Phi_\beta \Phi_\beta^{-1} = \frac{1}{2} (\partial_{zz} \phi_\beta / \partial_z \phi_\beta) \text{diag}[-1, 1]
\]

(3.78)

and is well behaved even at turning points, where \( \partial_{zz} \phi_\beta / \partial_z \phi_\beta = (\partial_{zz} \beta - 3p(\partial_z \beta)^2)/4\partial_z \beta \).

If then we construct the matrix \( C[p, z] \hat{E}(\omega, p, z) \) as an approximation to a fundamental matrix \( B \), this will be effective at high frequencies when \( \omega H_\beta \) is the dominant term in (3.77). Comparison with (3.59)–(3.60) shows that we have failed to represent the contributions from \( C^{-1} \partial_z C \) and \( \partial_z \Phi \Phi^{-1} \) and we will need to modify \( CE \) to allow for these effects.

We will first look at the character of the high frequency approximation and then discuss ways of improving on this solution.

At high frequencies the argument of the Airy functions becomes large and we may use the asymptotic representations of the functions. Below an \( S \) wave
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turning point in a perfectly elastic medium (i.e. \( q^2_\beta < 0 \)), the entries of \( \mathbf{E}_\beta \) are asymptotically

\[
\begin{align*}
A_j(\omega \tau_\beta) &\sim \frac{1}{2} p^{1/2} |q_\beta|^{-1/2} \exp(-\omega |\tau_\beta|), \\
A_k(\omega \tau_\beta) &\sim -\frac{1}{2} p^{-1/2} |q_\beta|^{1/2} \exp(-\omega |\tau_\beta|), \\
B_j(\omega \tau_\beta) &\sim p^{1/2} |q_\beta|^{-1/2} \exp(\omega |\tau_\beta|), \\
B_k(\omega \tau_\beta) &\sim -p^{-1/2} |q_\beta|^{1/2} \exp(\omega |\tau_\beta|).
\end{align*}
\]

(3.79)

In this region \( \mathbf{E}_\beta \) gives a good description of the evanescent wavefields. Above the turning point, the asymptotic behaviour of \( A_j \) and \( B_k \) is as \( \cos(\omega \tau_\beta - \pi/4) \), and for \( A_k \) and \( B_j \) as \( \sin(\omega \tau_\beta - \pi/4) \). These elements thus describe standing waves, but for most purposes we would prefer to consider travelling wave forms.

We can achieve this goal by constructing a new matrix \( \mathbf{E}_\beta \), whose columns are a linear combination of those of \( \mathbf{E}_\beta \),

\[
\mathbf{E}_\beta = \mathbf{E}_\beta.2^{-1/2} \begin{bmatrix} e^{-i\pi/4} & e^{i\pi/4} \\ e^{i\pi/4} & e^{-i\pi/4} \end{bmatrix}.
\]

(3.80)

This new matrix will also satisfy an equation of the form (3.77), and so \( \mathbf{C}(z)\mathbf{E}(z) \) is an equally good candidate for an approximate \( \mathbf{B} \) matrix at high frequencies. In terms of the Airy function entries

\[
\begin{align*}
\mathbf{E}_\beta &= 2^{-1/2} \begin{bmatrix} s_\beta e^{i\pi/4}(A_j - iB_j) & s_\beta e^{-i\pi/4}(A_j + iB_j) \\ e^{i\pi/4}(A_k - iB_k) & e^{-i\pi/4}(A_k + iB_k) \end{bmatrix} \\
&= 2^{-1/2} \begin{bmatrix} s_\beta E_j & s_\beta F_j \\ E_k & F_k \end{bmatrix}.
\end{align*}
\]

(3.81)

Once again at high frequencies we may use the asymptotic forms of the Airy functions and then, well above a turning point with \( q^2_\beta > 0 \):

\[
\begin{align*}
E_j(\omega \tau_\beta) &\sim (q_\beta/p)^{-1/2} \exp(i\omega \tau_\beta), \\
E_k(\omega \tau_\beta) &\sim -i(q_\beta/p)^{1/2} \exp(i\omega \tau_\beta), \\
F_j(\omega \tau_\beta) &\sim (q_\beta/p)^{-1/2} \exp(-i\omega \tau_\beta), \\
F_k(\omega \tau_\beta) &\sim i(q_\beta/p)^{1/2} \exp(-i\omega \tau_\beta).
\end{align*}
\]

(3.82)

With our convention that \( z \) increases downwards, \( \tau_\beta \) (3.73) is a decreasing function of \( z \) for \( z > Z_\beta \), the turning level for the \( S \) waves. We have adopted a time factor \( \exp(-i\omega t) \) so that asymptotically \( E_j, E_k \) have the character of upgoing waves; similarly \( F_j, F_k \) have the character of downgoing waves. These interpretations for large arguments \( \omega \tau_\beta \) are misleading if extrapolated too close to the turning point.

Below the turning point all the entries \( \mathbf{E}_\beta \) increase exponentially with depth because of the dominance of the \( B_i \) terms, and so \( \mathbf{E}_\beta \) is not useful in the evanescent
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regime. Once again $E_\beta$ has a simple inverse which may be expressed in terms of the entries of $E_\beta$:

$$E_\beta^{-1} = i2^{-1/2} \begin{bmatrix} -s_\beta Fk & Fj \\ s_\beta Ek & -Ej \end{bmatrix}. \tag{3.83}$$

In the propagating regime, above a turning point we will use $E_\beta$ as the basis for our fundamental $B$ matrix, and in the evanescent region we will use $\hat{E}_\beta$. Both of these matrices were constructed to take advantage of the uniform approximations afforded by the Airy function for an isolated turning point. A different choice of phase matrix with parabolic cylinder function entries is needed for a uniform approximation with two close turning points (Woodhouse, 1978).

For the $P$-$SV$ wave system we have constructed $C_P$ to give a high frequency block diagonal structure (3.63) and so the corresponding phase matrix $E$ has a block diagonal form. Above all turning points $E$ has $\hat{E}_\alpha$ for the $P$ wave contribution, and $\hat{E}_\beta$ for the $SV$ contribution. Below both $P$ and $S$ turning levels $\hat{E}$ is constructed from $\hat{E}_\alpha$, $\hat{E}_\beta$. In the intermediate zone below the $P$ turning level, so that $P$ waves are evanescent, but with $S$ waves still propagating, we take the block diagonal form

$$\hat{E} = \begin{bmatrix} \hat{E}_\alpha & 0 \\ 0 & \hat{E}_\beta \end{bmatrix}. \tag{3.84}$$

In the high frequency limit we ignore any coupling between $P$ and $SV$ waves.

We now wish to improve on our high frequency approximations to the fundamental stress-displacement matrix which we will represent as

$$B_0(\omega, p, z) = C(p, z)E(\omega, p, z), \tag{3.85}$$

in terms of some phase matrix representing the physical situation. As in our discussion of the eigenvector decomposition we want to retain the merits of $E$ in representing phase terms, so we look for correction matrices which either pre- or post-multiply $E$.

### 3.3.1 An asymptotic expansion

If we look for a fundamental $B$ matrix in the form

$$B(\omega, p, z) = C(p, z)K(\omega, p, z)E(\omega, p, z), \tag{3.86}$$

then our correction matrix $K$ must satisfy

$$\partial_\omega K = \omega[HK - KH] - C^{-1}\partial_\omega CK - K\partial_\omega \Phi \Phi^{-1}, \tag{3.87}$$

where we have used (3.59) and the differential equation for $E$ (3.77). The matrix $K$ is independent of the choice for $E$ and the frequency $\omega$ enters only through
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the commutator term, this suggests an asymptotic expansion in inverse powers of frequency,

\[ K(z) = I + \sum_{r=1}^{\infty} \omega^{-r}k_r(z). \] (3.88)

On substituting (3.88) into (3.87) and equating powers of \( \omega \) we obtain the recursive equations

\[ [Hk_{r+1} - k_{r+1}H] = -\partial_z k_r + C^{-1}\partial_z Ck_r + k_r\partial_z \Phi \Phi^{-1}, \] (3.89)

for the \( k_r \), starting with \( k_0 = 0 \). These equations have been solved by Woodhouse (1978) who has given detailed results for the coefficient \( k_1(z) \).

The fundamental stress-displacement matrix \( B \) has an asymptotic form, to first order,

\[ B_A(\omega, p, z) \sim C(p, z)[I + \omega^{-1}k_1(p, z)]E(\omega, p, z), \] (3.90)

where we have shown the explicit dependence of the various terms on frequency, slowness and depth. A merit of this approach is that the correction \( k_1 \) is independent of frequency. The propagator for a region \((z_A, z_B)\) can be constructed from \( B_A \) as

\[ P(z_A, z_B) \sim C(z_A)[I + \omega^{-1}k_1(z_A)]E(z_A)E^{-1}(z_B)[I - \omega^{-1}k_1(z_B)]C^{-1}(z_B), \] (3.91)

to first order. The representation (3.91) shows no identifiable reflections within \((z_A, z_B)\) and this makes it difficult to gain any insight into interactions between the wavefield and parameter gradients. This asymptotic form of propagator has mostly been used in surface wave and normal mode studies (see, e.g., Kennett & Woodhouse, 1978; Kennett & Nolet, 1979).

### 3.3.2 Interaction series

An alternative scheme for constructing a full \( B \) matrix is to postmultiply the high frequency approximation by \( L(z) \) so that

\[ B(\omega, p, z) = C(p, z)\mathbf{E}(\omega, p, z)L(\omega, p, z). \] (3.92)

The matrix \( L \) then satisfies

\[ \partial_z L = -\mathbf{E}^{-1}[C^{-1}\partial_z C + \partial_z \Phi \Phi^{-1}]EL = \{\mathbf{E}^{-1}\mathbf{E}\}L. \] (3.93)

The frequency dependence of \( L \) arises from the phase terms \( \mathbf{E}, \mathbf{E}^{-1} \) and the form of \( L \) is controlled by the choice of \( \mathbf{E} \). The equation for \( L \) is equivalent to the integral equation

\[ L(z; z_{\text{ref}}) = I + \int_{z_{\text{ref}}}^{z} d\zeta \{\mathbf{E}^{-1}(\zeta)\mathbf{j}(\zeta)\mathbf{E}(\zeta)\}L(\zeta), \] (3.94)
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where we would choose the lower limit of integration \( z_{ref} \) to correspond to the physical situation. Thus for a region containing a turning point we take \( z_{ref} \) at the turning level and use different forms of the phase matrix above and below this level. Otherwise we may take any convenient reference level. We may now make an iterative solution of (3.94) (Chapman 1981, Kennett & Illingworth 1981) in terms of an ‘interaction series’. We construct successive estimates as

\[
L_r(z; z_{ref}) = I + \int_{z_{ref}}^{z} d\zeta \left \{ E^{-1}(\zeta)[j(\zeta)E(\zeta)]L_{r-1}(\zeta) \right \}, \tag{3.95}
\]

with \( L_0 = I \). At each stage we introduce a further coupling into the parameter gradients present in \( j \) through \( C^{-1} \partial_z C \) and \( \partial_z \Phi \Phi \Phi \Phi \Phi \Phi \Phi^{-1} \). All the elements of \( E^{-1}jE \) are bounded and so the recursion (3.95) will converge. The interaction series is therefore of the form

\[
L(z; z_{ref}) = I + \int_{z_{ref}}^{z} d\zeta \left \{ E^{-1}(\zeta)[j(\zeta)E(\zeta)] \int_{z_{ref}}^{\zeta} d\eta \left \{ E^{-1}(\eta)jE(\eta) \right \} \right \} + \ldots \tag{3.96}
\]

and the terms may be identified with successive interactions of the seismic waves with the parameter gradients. For slowly varying media the elements of \( j \) will be small, and the series (3.96) will converge rapidly. In this case it may be sufficient to retain only the first integral term.

For \( SH \) waves the total gradient effects are controlled by the diagonal matrix

\[
j_H = \left \{ \frac{1}{2} \partial_z \mu / \mu + \frac{1}{2} \partial_{zz} \phi_\beta / \partial_z \phi_\beta \right \} \text{diag}[1, -1], \tag{3.97}
\]

and

\[
E^{-1}_\beta j_H E_\beta = -\frac{i}{4}(\partial_z \mu / \mu + \partial_{zz} \phi_\beta / \partial_z \phi_\beta) \times \left \{ E j_\beta F_k \beta + E k_\beta F_j \beta \begin{bmatrix} 2F_j \beta F_k \beta \\ -2E j_\beta E k_\beta \\ -(E j_\beta F_k \beta + E k_\beta F_j \beta) \end{bmatrix} \right \}, \tag{3.98}
\]

where we have written \( E j_\beta \) for \( E j(\omega \tau_\beta) \). Asymptotically, well away from any turning level, in the propagating and evanescent regimes

\[
\partial_{zz} \phi_\beta / \partial_z \phi_\beta \approx \partial_z q_\beta / q_\beta, \tag{3.99}
\]

and then the gradient term behaves like \( \gamma_H \) (3.46). In the propagating regime, for example,

\[
E^{-1}_\beta j_H E_\beta \approx \gamma_H \begin{bmatrix} 0 & \exp[-2i\omega \tau_\beta] \\ \exp[+2i\omega \tau_\beta] & 0 \end{bmatrix}, \tag{3.100}
\]

and the interaction series parallels the treatment for the eigenvector decomposition. Now, however, (3.98) is well behaved at a turning level.

For the \( P-SV \) system, the structure of the block diagonal entries of \( j_P \) for \( P \) and \( S \) parallel our discussion for the \( SH \) wave case and asymptotically the coefficients...
γ_P, γ_S control the interaction with the gradients. The off-diagonal matrices lead to interconversion of P and SV waves and here asymptotically we recover γ_T and γ_R determining transmission and reflection effects.

The structure of the interaction terms L for P-SV waves is

\[ L_P = \begin{bmatrix} I + L_{\alpha\alpha} & L_{\alpha\beta} \\ L_{\beta\alpha} & I + L_{\beta\beta} \end{bmatrix}. \] (3.101)

\( L_{\alpha\alpha}, L_{\beta\beta} \) represent multiple interactions with the parameter gradients without change of wave type. \( L_{\alpha\beta}, L_{\beta\alpha} \) allow for the coupling between \( P \) and \( SV \) waves that is not present in our choice of phase matrix and which only appears with the first integral contribution in (3.96).

From the series (3.96) we may find \( L(\omega, p, z; z_R) \) to any desired level of interaction with the medium and then construct an approximate \( B \) matrix as

\[ B_I(\omega, p, z) \approx C(z_A)E(\omega, p, z)L(z_A; z_{ref})L^{-1}(z_B; z_{ref})E^{-1}(z_B)C^{-1}(z_B), \] (3.102)

For a region \((z_A, z_B)\) the propagator may be approximated as

\[ P(z_A, z_B) \approx C(z_A)E(z_A)L(z_A; z_{ref})L^{-1}(z_B; z_{ref})E^{-1}(z_B) \] (3.103)

and the kernel \( L(z_A; z_{ref})L^{-1}(z_B; z_{ref}) \) may now be identified with internal reflections in \((z_A, z_B)\). When a turning point occurs within the region we can split the calculation above and below the turning level and then use the chain rule for the propagator.

The asymptotic series approach in the previous section leads to a solution which is most effective at high frequencies when only a few terms need be included. The interaction series approach is not restricted in its frequency coverage. Although our starting point is a high frequency approximation to the solution, this is compensated for by the presence of the same term in the kernel of the interaction series development. The number of terms required to get an adequate approximation of the wavefield depends on \(\{E^{-1}JE\} \) and thus on the size of the parameter gradients and the frequency. At low frequencies we need more terms in the interaction series to counteract the high frequency character of \(E\).

### 3.3.3 Relation to eigenvector decomposition

If we adopt the interaction series approach, all our fundamental matrix representations include the *leading order* approximation \( C(p, z)E(\omega, p, z) \). This form gives no coupling between \( P \) and \( S \) waves but such coupling will be introduced by the interaction term \( L(\omega, p, z) \).

At high frequencies, \( CE \) asymptotically comes to resemble the WKBJ solution (3.52), since the depth behaviour is the same although there may be constant amplitude and phase factors between the two forms. To exploit this relation we extend an idea of Richards (1976) and rearrange \( CE \) to a form with only diagonal phase terms.
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(a) Propagating forms

We introduce the generalized vertical slownesses

\[
i_\eta(u)(p,\omega,z) = -p E_k(\omega \tau_\beta) / E_j(\omega \tau_\beta),
\]

\[
i_\eta(d)(p,\omega,z) = p F_k(\omega \tau_\beta) / F_j(\omega \tau_\beta).
\]

(3.104)

These quantities depend on slowness \(p\) and frequency \(\omega\) through the Airy terms. Since \(\tau_\beta\) depends on the slowness structure up to the reference level, \(\eta_\beta\) is not just defined by the local elastic properties. We have used the subscripts \(u,d\) as opposed to \(U,D\) to indicate that the Airy elements have up and downgoing character only in the asymptotic regime, far from a turning level. In this asymptotic region, with \(q_2 > 0\)

\[
\eta_\beta(u)(p,\omega) \sim q_\beta(p), \quad \eta_\beta(d)(p,\omega) \sim q_\beta(p),
\]

(3.105)

and also

\[
(2\rho p)^{-1/2} E_j(\omega \tau_\beta) \sim \epsilon_\beta \exp[-i \omega \int_{z_\beta}^z d\zeta q_\beta(\zeta)].
\]

(3.106)

With the aid of the generalized slownesses \(\eta_{u,d}\) we can recast \(CE\) into a form where we use only \(Ej, Fj\) to describe the phase behaviour. Thus we can write, for \(SH\) waves

\[
C_H E_H = D_H E_H = [b_{H} u, b_{H} d] E_H,
\]

(3.107)

where \(E_H\) is the diagonal matrix

\[
E_H = (2\rho p)^{-1/2} \text{diag}[E_j(\omega \tau_\beta), F_j(\omega \tau_\beta)].
\]

(3.108)

The column vectors of \(D_H\) are given by

\[
b_{H} u,d = [s_\beta, -i \rho \beta \eta_{u,d}, s_\beta \rho(2\beta_2^2 - 1)]^T,
\]

(3.109)

where we take the upper sign with the suffix \(u\), and \(s_\beta = 1\) when the shear velocity increases with depth. The new way of writing the leading order approximation is designed to emphasise the connection with the WKBJ solution. At high frequencies we see from (3.105) that, asymptotically, the columns of \(D\) reduce to those of the eigenvector matrix \(D\), and the phase terms reduce to (3.52).

For \(P-SV\) waves we take

\[
C_P E_P = D_P E_P = [b_{P} u, b_{P} d, b_{S} u, b_{S} d] E_P,
\]

(3.110)

with

\[
E_P = (2\rho p)^{-1/2} \text{diag}[E_j(\omega \tau_\alpha), F_j(\omega \tau_\alpha), E_j(\omega \tau_\beta), F_j(\omega \tau_\beta)],
\]

(3.111)

and column vectors

\[
b_{P} u,d = [\mp i \eta_{au,d}, s_\alpha p, s_\alpha \rho(2\beta_2^2 p^2 - 1), \mp 2i \rho \beta^2 \eta_{au,d}]^T,
\]

\[
b_{S} u,d = [s_\beta p, \mp i \eta_{bu,d}, \mp 2i \rho \beta^2 \eta_{bu,d}, s_\beta \rho(2\beta_2^2 p^2 - 1)]^T.
\]

(3.112)
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Once again these expressions reduce to the WKBJ forms in the high frequency asymptotic limit.

(b) Evanescent forms
Below a turning point we work in terms of the Ai, Bi Airy functions and now take modified forms of the generalized vertical slownesses

\[ \begin{align*}
\tilde{\eta}_{\beta u} &= -pBk(\omega \tau_\beta)/Bj(\omega \tau_\beta), \\
\tilde{\eta}_{\beta d} &= pAk(\omega \tau_\beta)/Aj(\omega \tau_\beta).
\end{align*} \] (3.113)

At high frequency, in the far evanescent regime

\[ \begin{align*}
\tilde{\eta}_{\beta u} &\sim i|q_\beta|, \\
\tilde{\eta}_{\beta d} &\sim i|q_\beta|,
\end{align*} \] (3.114)

and now

\[ \begin{align*}
(\rho p)^{-1/2}A_j(\omega \tau_\beta) &\sim (i/2)^{1/2}e_\beta \exp[-\omega |\tau_\beta|], \\
(\rho p)^{-1/2}B_j(\omega \tau_\beta) &\sim (2i)^{1/2}e_\beta \exp[-\omega |\tau_\beta|].
\end{align*} \] (3.115) (3.116)

In this region we write the leading order approximation \( \mathbf{C} \mathbf{E} \) in a comparable form to the propagating case

\[ \mathbf{C}_H \tilde{\mathbf{E}}_H = \mathbf{D}_H \tilde{\mathbf{E}}_H = \mathbf{b}_u^H \mathbf{b}_d^H \tilde{\mathbf{E}}_H. \] (3.117)

The column vectors \( \mathbf{b}_d^H \) differ from \( \mathbf{b}_u^H \) by using the modified slownesses \( \tilde{\eta} \). The phase term \( \tilde{\mathbf{E}} \) has the same character as the WKBJ solution, but the overall amplitudes and phases differ.

For \( P-SV \) waves we make a similar development to the above when both \( P \) and \( S \) waves are evanescent. When only \( P \) waves are evanescent we use the \( \mathbf{b}_P \) forms and retain the propagating \( \mathbf{b}_S \) vectors for \( S \) waves.

The organisation of the fundamental \( \mathbf{B} \) matrix by wave type is very convenient for a discussion of turning point phenomena. When we come to consider reflection and transmission problems in Chapter 5, we shall see that an organisation by the asymptotic character of the column elements is preferable. The two fundamental matrices are related by a constant matrix multiplier and so a switch between the two forms is easily made.

For \( SH \) waves the fundamental matrix \( \mathbf{B}_I \) (3.103) is already organised into columns whose characters are asymptotically that of up and downgoing waves. For coupled \( P-SV \) waves the fundamental matrix \( \mathbf{B}_{ud} \) with this organisation is given by

\[ \mathbf{B}_{ud} = \Xi \mathbf{B}_I \] (3.118)

where

\[ \Xi = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}, \quad \Xi = \Xi^{-1}. \] (3.119)

The symmetric matrix \( \Xi \) achieves the desired reorganisation of the columns of \( \mathbf{B}_I \).
Appendix: Transverse isotropy

In a transversely isotropic medium the wavespeed in directions perpendicular to the symmetry axis are all the same but differ from those parallel to the axis. If the symmetry axis is vertical, the elastic properties do not vary in a horizontal plane. We may once again use an expansion in terms of vector harmonics as in (2.55) and the corresponding stress-displacement vectors satisfy first order differential equations (2.26) with coefficient matrices \( A(p, z) \) which depend only on the slowness \( p \) (Takeuchi & Saito, 1972).

With this form of symmetry there are five independent elastic moduli: \( A, C, F, L, N \). In the more restricted case of isotropy there are only two independent moduli \( \lambda, \mu \) and then

\[
A = C = \lambda + 2\mu, F = \lambda, L = N = \mu. \tag{3a.1}
\]

Thus \( A, C \) and \( F \) are related to dilatation waves and \( L, N \) to shear waves. In a homogeneous medium three types of plane waves exist. For horizontal transmission, the wavespeeds of \( P, SV \) and \( SH \) waves are

\[
\alpha_h = (A/\rho)^{1/2}, \beta_v = (L/\rho)^{1/2}, \beta_h = (N/\rho)^{1/2}. \tag{3a.2}
\]

For vertical transmission the wavespeeds are

\[
\alpha_v = (C/\rho)^{1/2}, \beta_v = (L/\rho)^{1/2}, \tag{3a.3}
\]

and there is no distinction between the \( SH \) and \( SV \) waves. In directions inclined to the \( z \)-axis, there are still three possible plane waves but their velocities depend on the inclination.

In a transversely isotropic medium where the properties depend on depth, there is a separation of the stress-displacement vector equations into \( P-SV \) and \( SH \) wave sets.

The coefficient matrix for \( P-SV \) waves is now

\[
A_p = \begin{bmatrix}
0 & \frac{pF}{C} & \frac{1}{C} & 0 \\
-p & 0 & 0 & \frac{1}{L} \\
-\rho & 0 & 0 & p \\
0 & -\rho + p^2(A - F^2/C) & -pF/C & 0 \\
\end{bmatrix}, \tag{3a.4}
\]

and its eigenvalues are \( \pm iq_1, \pm iq_2 \) where \( q \) is a root of

\[
q^4 + q^2G + H = 0, \tag{3a.5}
\]

with

\[
G = [\rho L + \rho C - \rho^2(AC - F^2 - 2FL)]/LC, \\
H = (\rho - \rho^2L)(\rho - \rho^2A)/LC, \tag{3a.6}
\]

and so

\[
q^2 = \frac{1}{2} [G \mp (G^2 - 4H)^{1/2}]. \tag{3a.7}
\]

In an isotropic medium the upper sign in the expression for \( q \) yields \( q_\alpha \), the lower sign \( q_\beta \). The eigenvector matrix \( D \) may be constructed from column vectors corresponding to up and downgoing quasi-\( P \) and \( S \) waves, and analysis parallels the isotropic case.

For \( SH \) waves the relations are rather simpler

\[
A_{sh} = \begin{bmatrix}
0 & 1/L \\
-\rho + p^2 & 0 \\
\end{bmatrix} = \begin{bmatrix}
0 & (\rho \beta_h^2 p^2 - 1)^{-1} \\
0 & \rho \beta_h^2 \\
\end{bmatrix}, \tag{3a.8}
\]

and its eigenvalues are \( \pm iq_h \), with

\[
q_h^2 = (\rho - \rho^2N)/L = (1 - \rho^2\beta_h^2)/\beta_h^2. \tag{3a.9}
\]
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which, except at vertical incidence \((p = 0)\), differs from the SV case. The main modification to our discussions of an isotropic medium is that the vertical slownesses for SV and SH are no longer equal.